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On the Ricci Tensor of Non-Stationary Axisymmetric Space-Times

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Abstract We recalculate the Ricci tensors of non-stationary axisymmetric space-times originally calculated by Chandrasekhar, and we find that in the linear regime there are some common factors that did not appear in the original results. We also find some discrepancies in the non-linear terms. However, these discrepancies do not affect the well-known results concerning linear perturbations of a Schwarzschild black hole.

Keywords Quasinormal modes · Schwarzschild black hole · Ricci tensor

1 Introduction

The theory of linear perturbations of a black hole is important in many aspects of modern general relativity. A perturbed black hole relaxes to its unperturbed state by emitting gravitational radiation with a characteristic signature (the normal mode), determined only by the mass and angular momentum of the black hole. It is expected that, following a compact object coalescence, advanced LIGO will be able to measure this ringdown effect [5]. Further, in a numerical relativity simulation, the gravitational radiation emitted is normally estimated by constructing a spherical shell of some fixed finite radius, and matching the computed solution to a linear perturbation on the shell [4, 7].

The theory of linear perturbations of a Schwarzschild black hole was first developed by Regge and Wheeler [6] and Zerilli [8]. The work was consolidated, and a relationship established between even [6] and odd [8] normal modes, by Chandrasekhar and others [1–3]. The starting point for the mathematical treatment of the subject is the Ricci tensor of a general non-stationary axisymmetric space-time. Formulas for $R_{\alpha\beta}$ are given in the classic text book [2] ((4) on pages 141–142), based on results originally given in [3]. These formulas

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were obtained by hand (at the time computer algebra did not exist) using the Cartan calculus. We have re-derived the formulas for $R_{\alpha\beta}$ using computer algebra (Maple), and found the original versions given in [3] and [2] are incorrect, both at the linear and nonlinear levels. However as discussed in Sect. 5, the errors do not affect the well known results concerning linear perturbations of a Schwarzschild black hole.

The plan of the paper is as follows, Sect. 2 presents our calculation of the Ricci tensor and the results. In Sect. 3 we investigated the differences between our results for $R_{\alpha\beta}$ and those of [2], at both the linear and nonlinear levels. Section 4 looks at the issue of justifying the correctness of our computer algebra script. We do so by constructing special cases in which hand calculation of $R_{\alpha\beta}$ is straightforward, and in which it is clear that the result is consistent with the formulas in Sect. 2, but not with those in [2]. We end with a conclusion, Sect. 5.

2 The Ricci Tensor for Non-Stationary Axisymmetric Space-Times

We follow the notation and sign conventions of Chandrasekhar [2], and consider the coordinates (t,ϕ,r,θ) for the line element of non-stationary axisymmetric space-times given by

$$ds^2 = e^{2\nu}(dt)^2 - e^{2\lambda}(d\phi - \omega dt - q_2 dx^2 - q_3 dx^3)^2 - e^{2\mu_2}(dx^2)^2 - e^{2\mu_3}(dx^3)^2, \qquad (1)$$

where ν, λ, ω, μ_2, μ_3, q_2, and q_3 are functions of $x^0 = t$, $x^2 = r$, and $x^3 = \theta$, but are independent of $x^1 = \phi$.

We found that

$$g^{02} = g^{03} = g^{23} = 0, \qquad (2)$$

$$g^{00} = e^{-2\nu}, \qquad g^{22} = -e^{-2\mu_2}, \qquad g^{33} = -e^{-2\mu_3}, \qquad (3)$$

$$g^{01} = \omega e^{-2\nu}, \qquad g^{12} = -q_2 e^{-2\mu_2}, \qquad g^{13} = -q_3 e^{-2\mu_3}, \qquad (4)$$

and

$$g^{11} = -e^{-2\lambda} + \omega^2 e^{-2\nu} - q_2^2 e^{-2\mu_2} - q_3^2 e^{-2\mu_3}. \qquad (5)$$

We have used computer algebra (Maple) to determine the Ricci tensor of the metric (1). We found

$$\begin{aligned}
R_{00} = {} & e^{-2\mu_3+2\lambda}\Big[-\omega^2(\lambda_{,\theta,\theta} + \lambda_{,\theta}(\lambda_{,\theta} + \mu_{2,\theta} - \mu_{3,\theta} + \nu_{,\theta})) \\
& + \omega(Q_{30,\theta} + Q_{30}(\mu_{2,\theta} - \nu_{,\theta} + 3\lambda_{,\theta} - \mu_{3,\theta})) - \frac{1}{2}Q_{30}^2\Big] \\
& + e^{-2\mu_2+2\lambda}\Big[-\omega^2(\lambda_{,r,r} - \lambda_{,r}(\lambda_{,r} + \mu_{3,r} - \mu_{2,r} + \nu_{,r})) \\
& + \omega(Q_{20,r} + Q_{20}(-\mu_{2,r} + \mu_{3,r} + 3\lambda_{,r} - \nu_{,r})) - \frac{1}{2}Q_{20}^2\Big] \\
& + e^{-2\nu+2\lambda}[\omega^2(\lambda_{,t,t} + \lambda_{,t}(\lambda_{,t} + \mu_{3,t} + \mu_{2,t} - \nu_{,t}))] \\
& + e^{-2\mu_3+2\nu}[\nu_{,\theta,\theta} + \nu_{,\theta}(\nu_{,\theta} + \lambda_{,\theta} + \mu_{2,\theta} - \mu_{3,\theta})] \\
& + e^{-2\mu_2+2\nu}[\nu_{,r,r} + \nu_{,r}(\nu_{,r} + \lambda_{,r} + \mu_{3,r} - \mu_{2,r})]
\end{aligned}$$

$$+\frac{1}{2}\omega^2 e^{4\lambda}[e^{-2\mu_2-2\mu_3}Q_{23}^2 - e^{-2\mu_3-2\nu}Q_{30}^2 - e^{-2\mu_2-2\nu}Q_{20}^2]$$
$$+[(\nu_{,t}-\mu_{3,t})\mu_{3,t}-\mu_{3,t,t}]+[(\nu_{,t}-\mu_{2,t})\mu_{2,t}-\mu_{2,t,t}]$$
$$+[(\nu_{,t}-\lambda_{,t})\lambda_{,t}-\lambda_{,t,t}], \tag{6}$$

$$\begin{aligned} R_{11} = {} & e^{-2\mu_3+2\lambda}[-\lambda_{,\theta,\theta}+\lambda_{,\theta}(-\lambda_{,\theta}-\mu_{2,\theta}-\nu_{,\theta}+\mu_{3,\theta})] \\ & + e^{-2\mu_2+2\lambda}[-\lambda_{,r,r}+\lambda_{,r}(-\lambda_{,r}-\nu_{,r}-\mu_{3,r}+\mu_{2,r})] \\ & + e^{-2\nu+2\lambda}[\lambda_{,t,t}+\lambda_{,t}(\lambda_{,t}+\mu_{2,t}+\mu_{3,t}-\nu_{,t})] \\ & + \frac{1}{2}e^{4\lambda}[e^{-2\mu_2-2\mu_3}Q_{23}^2 - e^{-2\mu_3-2\nu}Q_{30}^2 - e^{-2\mu_2-2\nu}Q_{20}^2], \end{aligned} \tag{7}$$

$$\begin{aligned} R_{22} = {} & e^{-2\mu_3+2\mu_2}[-\mu_{2,\theta,\theta}+\mu_{,\theta}(-\mu_{,\theta}-\nu_{,\theta}-\lambda_{,\theta}+\mu_{3,\theta})] \\ & + e^{-2\nu+2\mu_2}[\mu_{2,t,t}+\mu_{2,t}(\mu_{2,t}+\lambda_{,t}-\nu_{,t}+\mu_{3,t}+\mu_{2,t})] \\ & + e^{-2\mu_3+2\lambda}\Big[q_2(Q_{23}(-\nu_{,\theta}+\mu_{3,\theta}+\mu_{2,\theta}-3\lambda_{,\theta})-Q_{23,\theta}) \\ & + q_2^2(-\lambda_{,\theta,\theta}+\lambda_{,\theta}(-\lambda_{,\theta}+\mu_{3,\theta}-\nu_{,\theta}-\mu_{2,\theta}))-\frac{1}{2}Q_{23}^2\Big] \\ & + e^{-2\nu+2\lambda}\Big[q_2(Q_{20}(-\nu_{,t}-\mu_{2,t}+\mu_{3,t}+3\lambda_{,t})+Q_{20,t}) \\ & + q_2^2(\lambda_{,t,t}+\lambda_{,t}(\lambda_{,t}-\nu_{,t}+\mu_{3,t}+\mu_{2,t}))+\frac{1}{2}Q_{20}^2\Big] \\ & + e^{-2\mu_2+2\lambda}[q_2^2(-\lambda_{,r,r}+\lambda_{,r}(-\lambda_{,r}-\mu_{3,r}-\nu_{,r}+\mu_{2,r}))] \\ & + \frac{1}{2}e^{4\lambda}q_2^2[-e^{-2\mu_2-2\nu}Q_{20}^2 + e^{-2\mu_2-2\mu_3}Q_{23}^2 - e^{-2\mu_3-2\nu}Q_{30}^2] \\ & + [(\mu_{2,r}-\mu_{3,r})\mu_{3,r}-\mu_{3,r,r}]+[(\mu_{2,r}-\nu_{,r})\nu_{,r}-\nu_{,r,r}] \\ & + [(\mu_{2,r}-\lambda_{,r})\lambda_{,r}-\lambda_{,r,r}], \end{aligned} \tag{8}$$

$$\begin{aligned} R_{01} = {} & e^{-2\mu_2+2\lambda}\Big[\omega(\lambda_{,r,r}+\lambda_{,r}(\lambda_{,r}+\mu_{3,r}-\mu_{2,r}+\nu_{,r})) \\ & + \frac{1}{2}(-Q_{20,r}+Q_{20}(\nu_{,r}-3\lambda_{,r}+\mu_{2,r}-\mu_{3,r}))\Big] \\ & + e^{-2\nu+2\lambda}[\omega(-\lambda_{,t,t}+\lambda_{,t}(-\lambda_{,t}+\nu_{,t}-\mu_{2,t}-\mu_{3,t}))] \\ & + e^{-2\mu_3+2\lambda}\Big[\omega(\lambda_{,\theta,\theta}+\lambda_{,\theta}(\lambda_{,\theta}+\nu_{,\theta}+\mu_{2,\theta}-\mu_{3,\theta})) \\ & + \frac{1}{2}(-Q_{30,\theta}+Q_{30}(-3\lambda_{,\theta}+\mu_{3,\theta}-\mu_{2,\theta}+\nu_{,\theta}))\Big] \\ & + \frac{1}{2}e^{4\lambda}\omega[e^{-2\mu_3-2\nu}Q_{30}^2 + e^{-2\mu_2-2\nu}Q_{20}^2 - e^{-2\mu_2-2\mu_3}Q_{23}^2], \end{aligned} \tag{9}$$

$$R_{12} = \frac{1}{2}\Big[q_2(\lambda_{,\theta,\theta}+\lambda_{,\theta}(\lambda_{,\theta}+\nu_{,\theta}+\mu_{2,\theta}-\mu_{3,\theta}))$$

$$
\begin{aligned}
&+\frac{1}{2}(Q_{23,\theta}+Q_{23}(3\lambda_{,\theta}+\nu_{,\theta}-\mu_{2,\theta}-\mu_{3,\theta}))\Big]\\
&+e^{-2\nu+2\lambda}\Big[q_2(-\lambda_{,t,t}+\lambda_{,t}(-\lambda_{,t}+\nu_{,t}-\mu_{3,t}-\mu_{2,t}))\\
&+\frac{1}{2}(-Q_{20,t}+Q_{20}(\mu_{2,t}-3\lambda_{,t}-\mu_{3,t}+\nu_{,t}))\Big]\\
&+e^{-2\mu_2+2\lambda}[q_2(\lambda_{,r,r}+\lambda_{,r}(\lambda_{,r}-\mu_{2,r}+\nu_{,r}+\mu_{3,r}))]\\
&+\frac{1}{2}e^{4\lambda}q_2[e^{-2\mu_3-2\nu}Q_{30}^2-e^{-2\mu_2-2\mu_3}Q_{23}^2+e^{-2\mu_2-2\nu}Q_{20}^2],
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
R_{02}={}&e^{-2\nu+2\lambda}\Big[q_2\omega(\lambda_{,t,t}+\lambda_{,t}(\lambda_{,t}-\nu_{,t}+\mu_{2,t}+\mu_{3,t}))\\
&+\frac{1}{2}\omega(Q_{20,t}+Q_{20}(-\mu_{3,t}+3\lambda_{,t}-\nu_{,t}+\mu_{3,t}))\Big]\\
&+e^{-2\mu_2+2\lambda}\Big[q_2\omega(-\lambda_{,r,r}+\lambda_{,r}(-\lambda_{,r}+\mu_{2,r}-\mu_{3,r}-\nu_{,r}))\\
&+\frac{1}{2}q_2(Q_{20,r}+Q_{20}(3\lambda_{,r}+\mu_{3,r}-\nu_{,r}-\mu_{2,r}))\Big]\\
&+e^{-2\mu_3+2\lambda}\Big[\omega q_2(-\lambda_{,\theta,\theta}+\lambda_{,\theta}(-\lambda_{,\theta}+\mu_{3,\theta}-\nu_{,\theta}-\mu_{2,\theta}))\\
&+\frac{1}{2}Q_{30}(3\lambda_{,\theta}-\mu_{3,\theta}+\mu_{2,\theta}-\nu_{,\theta})\\
&+\frac{1}{2}\omega Q_{23}(\mu_{3,\theta}+\mu_{2,\theta}-\nu_{,\theta}-3\lambda_{,\theta})\\
&+\frac{1}{2}(q_2Q_{30,\theta}-\omega Q_{23,\theta})+\frac{1}{2}Q_{30}Q_{23}\Big]\\
&+\frac{1}{2}e^{4\lambda}q_2\omega[-e^{-2\mu_3-2\nu}Q_{30}^2+e^{-2\mu_2-2\mu_3}Q_{23}^2-e^{-2\mu_2-2\nu}Q_{20}^2]\\
&+\nu_{,r}(\mu_{3,t}+\lambda_{,t})+\mu_{2,t}(\lambda_{,r}+\mu_{3,r})-(\mu_{3,t,r}+\mu_{3,t}\mu_{3,r})\\
&-(\lambda_{,t,r}+\lambda_{,r}\lambda_{,t}),
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
R_{23}={}&e^{-2\nu+2\lambda}\Big[q_2q_3(\lambda_{,t,t}+\lambda_{,t}(\lambda_{,t}-\nu_{,t}+\mu_{3,t}+\mu_{2,t}))\\
&+\frac{1}{2}q_3Q_{20}(-\mu_{2,t}+3\lambda_{,t}+\mu_{3,t}-\nu_{,t})\\
&+\frac{1}{2}q_2Q_{30}(3\lambda_{,t}-\mu_{3,t}+\mu_{2,t}-\nu_{,t})\\
&+\frac{1}{2}q_2(Q_{30,t}+Q_{20,t})+\frac{1}{2}Q_{20}Q_{30}\Big]\\
&+e^{-2\mu_2+2\lambda}\Big[q_2q_3(-\lambda_{,r,r}+\lambda_{,r}(-\lambda_{,r}+\mu_{3,r}-\nu_{,r}+\mu_{3,r}))
\end{aligned}
$$

$$+\frac{1}{2}q_2Q_{23}(3\lambda_{,r}+\lambda_{,r}-\mu_{2,r}-\mu_{3,r})\Big]$$
$$+e^{-2\mu_3+2\lambda}\Big[q_2q_3(-\lambda_{,\theta,\theta}+\lambda_{,\theta}(-\lambda_{,\theta}-\nu_{,\theta}+\mu_{3,\theta}+\mu_{2,\theta}))$$
$$+\frac{1}{2}q_3Q_{23}(\mu_{2,\theta}-3\lambda_{,\theta}+\mu_{3,\theta}-\nu_{,\theta})-\frac{1}{2}q_3Q_{23,\theta}\Big]$$
$$+(\nu_{,\theta}+\lambda_{,\theta})\mu_{3,\theta}+(\lambda_{,r}+\nu_{,r})\mu_{2,r}-(\lambda_{,\theta,r}+\lambda_{,r}\lambda_{,\theta})$$
$$-(\nu_{,\theta,r}+\nu_{,\theta}\nu_{,r}), \tag{12}$$

where $Q_{AB}=q_{A,B}-q_{B,A}$, $Q_{A0}=q_{A,0}-\omega_{,A}$ $(A,B=2,3)$. As noted in [2], the remaining components R_{03}, R_{13}, and R_{33}, may be obtained from R_{02}, R_{12} and R_{22} respectively, by means of interchanging indices $2\leftrightarrow 3$ and $_{,\theta}\leftrightarrow {}_{,r}$.

3 Comparison Between the Ricci Tensor Reported Here, and the Expressions Given in [2]

3.1 The Linearized Case

The Schwarzschild solution is a special case of (1) where

$$e^{2\nu}=e^{-2\mu_2}=1-2M/r=\Delta/r^2,\qquad e^{\mu_3}=r,\qquad e^{\lambda}=r\sin\theta, \tag{13}$$

and

$$\omega=q_2=q_3=0\quad (x^0=t,\ x^1=\phi,\ x^2=r,\ x^3=\theta). \tag{14}$$

We now linearize about Schwarzschild, which means that we write

$$\nu=\nu_s+\delta\nu,\qquad \lambda=\lambda_s+\delta\lambda,\qquad \mu_2=\mu_{2s}+\delta\mu_2,\qquad \mu_3=\mu_{3s}+\delta\mu_3, \tag{15}$$

and regard ω, q_2, q_3, $\delta\nu$, $\delta\lambda$, $\delta\mu_2$, $\delta\mu_3$ as small quantities and neglect any term that involves the product of two or more small quantities. We compared the linearized Ricci tensors reported in (6) to (12) with those of [2], and found that in each case the relationship is of the form of a common factor. More precisely, defining

$$f_{\alpha\beta}=\frac{\text{linearized value of Ricci tensor reported by Chandrasekhar}}{\text{linearized value of Ricci tensor found by Maple program}} \tag{16}$$

we found the values presented in Table 1.

Table 1 The common factors $f_{\alpha\beta}$, as defined in (16)

$\alpha\beta$	$f_{\alpha\beta}$	$\alpha\beta$	$f_{\alpha\beta}$	$\alpha\beta$	$f_{\alpha\beta}$
00	$e^{-2\nu}$	33	$-e^{-2\mu_3}$	13	$e^{-\lambda-\mu_3}$
11	$-e^{-2\lambda}$	01	$e^{-\nu-\lambda}$	12	$e^{-\mu_2-\lambda}$
22	$e^{-2\mu_2}$	02	$-e^{-\mu_2-\nu}$	23	$-e^{-\mu_3-\mu_2}$

3.2 Nonlinear Terms

The results obtained in the linearized case suggest that we should investigate whether the full Ricci tensor found here and in [2] are related by the common factors given in Table 1. This turns out not to be the case. We proceed by evaluating

$$N_{\alpha\beta} = f_{\alpha\beta}(R_{\alpha\beta} \text{ found here}) - (R_{\alpha\beta} \text{ found in [2]}). \tag{17}$$

Should the hypothesis that the Ricci tensors are related by a common factor be true, then we would find $N_{\alpha\beta} = 0$. However, the expressions were nonzero, and indeed were not simple. As an illustration we give the value of $N_{\alpha\beta}$ in one case:

$$\begin{aligned} N_{00} = 1/2\omega\big[&e^{-4\nu+2\lambda}\{2\omega(\lambda_{,t,t} + \lambda_{,t}(\lambda + \mu_3 + \mu_2 - \nu)_{,t})\} \\ &+ e^{-2\nu-2\mu_2+2\lambda}\{-2\omega(\lambda_{,r,r} + \lambda_{,r}(\lambda + \mu_3 - \mu_2 + \nu)_{,r}) \\ &+ 2Q_{20}(\mu_3 - \mu_2 - \nu + 3\lambda)_{,r} + 2Q_{20,r}\} \\ &+ e^{-2\nu-2\mu_3+2\lambda}\{-2\omega(\lambda_{,\theta,\theta} + \lambda_{,\theta}(\lambda + \mu_2 + \nu - \mu_3)_{,\theta}) \\ &+ 2Q_{30}(-\mu_3 + \mu_2 + 3\lambda - \nu)_{,\theta} + 2Q_{30,\theta}\} \\ &- \omega Q_{20}^2 e^{-4\nu+4\lambda-2\mu_2} - \omega Q_{30}^2 e^{-4\nu-2\mu_3+4\lambda} + \omega Q_{23}^2 e^{-2\nu-2\mu_2+4\lambda-2\mu_3}\big]. \end{aligned} \tag{18}$$

4 Validation

Obtaining the expressions (6) to (12) for the Ricci tensor of the metric (1) is a lengthy process that requires the use of computer algebra. We can validate our results by constructing a special case in which the evaluation of the Ricci tensor is sufficiently simple that it can be done manually. In that way, we prove that the expressions in [2] are incorrect, and demonstrate self-consistency between the computer algebra and manual results.

Case 1: Suppose that

$$\lambda = \lambda(t) \quad \text{and} \quad \omega, \nu, \mu_2, \mu_3, q_2, q_3 \text{ all } 0. \tag{19}$$

Then the only non-zero components of the metric connection are $\Gamma^0_{11} = e^{2\lambda}\lambda_{,t}$, $\Gamma^1_{10} = \lambda_{,t}$. Now, the Ricci tensor component

$$R_{11} = e^{2\lambda}(\lambda_{,t,t} + \lambda^2_{,t}), \tag{20}$$

which is consistent with (7) as well as the 11 entry in Table 1, thus demonstrating the correctness of our results in the linearized case.

Case 2: From Sect. 3.2 we suppose that

$$\lambda = \lambda(t), \quad \omega = \text{constant}, \quad \text{and} \quad \nu, \mu_2, \mu_3, q_2, q_3 \text{ all } 0. \tag{21}$$

Then the only non-zero components of the metric connection are

$$\begin{aligned}
\Gamma^0_{00} &= e^{2\lambda}\omega^2\lambda_{,t},\\
\Gamma^0_{01} &= -e^{2\lambda}\omega\lambda_{,t},\\
\Gamma^1_{00} &= (-2+e^{2\lambda}\omega^2)\omega\lambda_{,t},\\
\Gamma^0_{11} &= e^{2\lambda}\lambda_{,t},\\
\Gamma^1_{01} &= -(-1+e^{2\lambda}\omega^2)\lambda_{,t},\\
\Gamma^1_{11} &= e^{2\lambda}\omega\lambda_{,t}.
\end{aligned} \tag{22}$$

Now, the Ricci tensor component

$$R_{00} = (\lambda^2_{,t} + \lambda_{,t,t})e^{2\lambda}\omega^2 - (\lambda^2_{,t} + \lambda_{,t,t}), \tag{23}$$

which is consistent with (6). However, R_{00} in [2], and subject to the simplifications in (21), is

$$R_{00} = -\lambda_{,t,t} - \lambda^2_{,t}. \tag{24}$$

This demonstrates that, at the nonlinear level, terms are missing from the results given in [2].

5 Conclusion

We have used computer algebra to construct the Ricci tensor of a general non-stationary axisymmetric space-time. We have compared our results to those previously reported, and found differences at both the linear and nonlinear levels. By considering a special case, we were able to demonstrate consistency in our results, and also that the results in [2] are incorrect.

The expression for the Ricci tensor of metric (1) is important, because it is the starting point for the whole theory of linear perturbations of a black hole, including the Zerilli and Regge-Wheeler equations. However, the errors in previous work reported here, do not affect this theory. The reasons are that in the case of linear perturbations the errors are in the form of a common factor, and a black hole space-time is vacuum so the common factors just cancel out.

Work that uses the Ricci tensor of [2] for a study of non-vacuum linear perturbations, or in the nonlinear regime, may need to be re-checked to ensure its correctness.

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